HEREDITARY TOPOLOGICAL CATEGORIES AND APPLICATIONS TO CLASSES OF CONVERGENCE SPACES

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Abstract

The paper gives a summary and improvements of results by the first author on hereditary topological categories and applies them to some familiar categories of convergence spaces.

1980 AMS Subject classification: 54B30, 54A05, 18B25

1. Introduction

The objective to distil the essential properties of categories considered by topologists and to unify their investigation lead to the introduction of topological categories (see [He 74], e.g.), i.e., small-fibred categories of structured sets and structure-compatible maps which have uniquely determined initial structures and where the empty set and the singletons carry only one structure. In order to develop a satisfactory concept of differential calculus, functional analysis, homotopy theory, and topological algebra in topological categories, additional "convenience" properties have been found desirable. The strongest of these properties is that the category be a topological universe:

1.1 Definition [Ne 84: 2.0]. A topological category is called a topological universe iff final epi-sinks are preserved by pullbacks along arbitrary morphisms.

There is also a categorial motivation for studying topological universes: They are exactly those topological categories (in the above sense) that are quasitopoi in the sense of Penon [Pe 73], [Pe 77: 4.8]. For details on this connection, see [Sc 86: Section 2], e.g. We will here not investigate the categorial aspects, however.

The concept of a topological universe splits naturally into two simpler ones: A topological category is a topological universe iff it is cartesian closed and hereditary. Recall that a category $\underline{A}$ with finite products is

Proceedings of the International Topology Conference (Baku, U.S.S.R., October 3-8, 1987), Baku 1989, 247-256

cartesian closed iff for each object $X \in \underline{A}$, the functor $- \times X : \underline{A} \longrightarrow \underline{A}$ has a right adjoint, which is equivalent to the existence of function spaces fulfilling an exponential law if $\underline{A}$ is topological.

While the significance of cartesian closedness has been recognized for several decades, and there are extensive investigations, the concept of heredity has only recently been discussed.

1.2 Definition [He 86]. A topological category is called hereditary iff quotients and coproducts are preserved by pullbacks along embeddings (i.e., for any quotient $f : X \longrightarrow Y$ and any subspace Z of Y, the restriction $f \restriction f^{-1}(Z) : f^{-1}(Z) \longrightarrow Z$ is a quotient; and for any coproduct $\coprod X_i$ and any subspace Z of $\coprod X_i$, $Z = \coprod_i (X_i \cap Z)$).

Section 2 contains a summary and some improvements of results on heredity in topological categories from [Sc 86]. In particular, we give some characterizations of hereditary topological categories, and conditions for the heredity of subcategories of hereditary topological categories. Surprisingly, our results show strong similarities to results on cartesian closedness; e.g., one-point-extensions play a similar role for heredity as function spaces for cartesian closedness.

The category $\underline{Top}$ of topological spaces and continuous maps is not hereditary. Indeed, the only topological subcategories of $\underline{Top}$ that are hereditary are the discrete and the indiscrete spaces [He 83: Theorem 2]. Even generalization to full subcategories of $\underline{Top}$ which are closed under formation of subspaces does not much to improve the situation: such categories are hereditary (in a slightly generalized sense) if and only if they consist either of discrete or of indiscrete spaces [He 87: 3.2]. Consequently, if looking for hereditary categories as replacements for $\underline{Top}$, one has to consider supercategories. Natural candidates are categories of convergence spaces. Applications of the theory of Section 2 to categories of convergence spaces are sketched in Section 3.

2. Heredity in topological categories

Before we can formulate a first characterization of hereditary topological categories, we need one more definition.

2.1 Definition. Let $\underline{A}$ be a topological category.

(1) A morphism from a subspace of $X \in \underline{A}$ to $Y \in \underline{A}$ is called a partial morphism from X to Y.

(2) An $\underline{A}$-object $Y^{\#}$ is a one-point-extension of Y iff Y can be embedded into $Y^{\#}$ by addition of a single point ∞_Y, and for every partial morphism $f : Z \longrightarrow Y$ from an object X to Y, the map $f^X : X \longrightarrow Y^{\#}$ defined by $f^X(x) = f(x)$ if $x \in Z$, $f^X(x) = \infty_Y$ if $x \notin Z$, is a morphism. If Y has a one-point-extension, we say that partial morphisms to Y are representable.

(3) If partial morphisms to all $\underline{A}$-objects are representable, we simply say that partial morphisms in $\underline{A}$ are representable.

2.2 Theorem [He 86: Theorem 1]. A topological category is hereditary iff partial morphisms are representable.

From 2.2 it is clear that one-point-extensions play an important role for the heredity of a topological category. It is therefore desirable to have concrete and handy descriptions of one-point-extensions. Two illuminative descriptions are given in 2.3 and 2.4. Recall that every topological category is partially ordered by defining $X \leq X'$ iff $|X| = |X'|$ and the identity map is a morphism from X to X' (where $|X|$ denotes the underlying set of the object X).

2.3 Proposition [Sc 86: 2.6]. If $\underline{A}$ is a hereditary topological category and $Y \in \underline{A}$, then $Y^{\#}$ is the greatest $\underline{A}$-object with underlying set $|Y| \cup \{\infty_Y\}$ which contains Y as a subspace. In particular, $Y^{\#}$ is uniquely determined (up to isomorphism).

If $\underline{A}$ is not hereditary, there need not always be a greatest $\underline{A}$-object with underlying set $|Y| \cup \{\infty_Y\}$ into which Y can be embedded (3.4(2)); even if these greatest objects exist for every $Y \in \underline{A}$, $\underline{A}$ need not be hereditary (3.5).

Recall that a subclass $\underline{D}$ of $\underline{A}$ is said to be finally dense in $\underline{A}$ iff for every $\underline{A}$-object Y, the $\underline{A}$-morphisms from $\underline{D}$-objects to Y form a final sink. (Initially dense subclasses are defined dually.)

2.4 Proposition [Sc 86: 2.10, 2.9]. If $\underline{A}$ is a hereditary topological category then for any $Y \in \underline{A}$ and for any finally dense subclass $\underline{D}$ of $\underline{A}$ (in particular, $\underline{D} = \underline{A}$), the sink
$(f^X : X \longrightarrow Y^{\#} \mid X \in \underline{D},\ f : Z \longrightarrow Y$ partial morphism from X to $Y)$ is a

final epi-sink.

While in a hereditary topological category $\underline{A}$, by 2.3, $Y^{\#}$ is bounded from above, being the greatest object (with underlying set $|Y| \cup \{\infty_Y\}$) with the property that Y is a subspace, 2.4 shows that $Y^{\#}$ is also bounded from below: it is the smallest object making all extensions of partial morphisms to Y $\underline{A}$-morphisms. Notice the strong analogy to greatest conjoining and smallest splitting structures for function spaces. It carries even further: greatest conjoining structures need not always exist; smallest splitting structures exist in every topological category and provide an endo-functor. Analogously, the finality construction of 2.4 can be carried out in every topological category, whether it is hereditary or not, and yields an endo-functor.

2.5 Definition [Sc 86: 2.8]. For any object Y of a topological category $\underline{A}$, we denote by $[Y]$ (or $[Y]_{\underline{A}}$, if necessary to avoid ambiguities) the $\underline{A}$-object endowed with the final $\underline{A}$-structure with respect to the family $(f^X : X \longrightarrow |Y| \cup \{\infty_Y\} \mid X \in \underline{A},\ f : Z \longrightarrow Y$ partial morphism from X to $Y)$.

For hereditary topological categories, we will use the notations $[Y]$ and $Y^{\#}$ interchangeably (even when not explicitly referring to the finality construction of 2.5).

We can now formulate more characterizations for the heredity of a topological category $\underline{A}$. Notice that 2.6(3) provides a particularly simple check if $\underline{A}$ has a very small or very simple initially dense subclass.– Again, we would like to point out the similarities to characterizations of cartesian closedness by means of splitting and conjoining structures (compare with [Sc 83a: 3.1, 3.2] or [Sc 83b: 2.2.10]).

2.6 Theorem [Sc 86: 2.11]. Let $\underline{D}$ be an initially dense subclass of the topological category $\underline{A}$. The following conditions are equivalent:

(1) $\underline{A}$ is hereditary.

(2) For any $Y \in \underline{D}$, partial morphisms to Y are representable.

(3) For any $Y \in \underline{D}$, the inclusion map $j : Y \longrightarrow [Y]$ is an embedding.

(4) For any $Y \in \underline{D}$, there is a greatest object with underlying set $|Y| \cup \{\infty_Y\}$ which contains Y as a subspace, and this object coincides with $[Y]$.

We conclude this section with conditions for subcategories of hereditary topological categories to be hereditary. We consider the cases of bireflective and bicoreflective subcategories. Analogous results concerning cartesian closedness may be found in [Sc 83b: 2.2.11, 2.3.7, 2.3.9, 2.3.16, 2.3.17].

2.7 Theorem [Sc 86: 3.1, 3.4], [Sc 87]. Let $\underline{A}$ be a hereditary topological category, $\underline{B}$ a bireflective subcategory of $\underline{A}$, and $\underline{D}$ an initially dense subclass of $\underline{B}$. Then the following conditions are equivalent:

(1) $\underline{B}$ is hereditary, and one-point-extensions in $\underline{B}$ are formed as in $\underline{A}$.

(2) $\underline{B}$ is closed with respect to one-point-extensions of $\underline{D}$-objects in $\underline{A}$ (i.e., $[Y]_{\underline{A}} \in \underline{B}$ for all $Y \in \underline{D}$).

(3) The reflector from $\underline{A}$ to $\underline{B}$ preserves subspaces of the form $Y \hookrightarrow [Y]_{\underline{A}}$, $Y \in \underline{D}$.

(4) The reflector from $\underline{A}$ to $\underline{B}$ preserves subspaces.

If $\underline{B}$ is, in addition, finally dense in $\underline{A}$, then the above conditions are equivalent to:

(5) $\underline{B}$ is hereditary.

Of course, the implication (1)$\Rightarrow$(5) in 2.7 holds for every bireflective subcategory $\underline{B}$ of a hereditary topological category $\underline{A}$. Without any additional condition, the converse is not true (3.3(2)). Final density of $\underline{B}$ is sufficient, but not necessary for the equivalence (1) $\Leftrightarrow$ (5) (see 3.3(1)).

2.8 Theorem [Sc 86: 4.1, 4.3]. Let $\underline{A}$ be a hereditary topological category and $\underline{B}$ a bicoreflective subcategory of $\underline{A}$ which is closed under formation of subspaces in $\underline{A}$. Then $\underline{B}$ is hereditary, and the one-point-extensions in $\underline{B}$ are obtained as the coreflective modifications of the one-point-extensions in $\underline{A}$. In particular, this holds if $\underline{B}$ is bireflective as well as bicoreflective in $\underline{A}$.

3. Applications to categories of convergence spaces

We will now illustrate possibilities to use the results of Section 2 by applying them to some familiar categories of convergence spaces.

Since, unfortunately, the notation is not quite settled, we start, for convenience of the reader, by recording the definitions. For references on convergence concepts, we refer to [Sc 83a] and [Sc 79], e.g.

3.1 Definition. A convergence space Y is a set equipped with a function which assigns to each point $x \in Y$ certain filters $\mathfrak{F}$ (not necessarily proper), called the filters converging to X and written $\mathfrak{F} \xrightarrow[Y]{} x$ or simply $\mathfrak{F} \longrightarrow x$, subject to the following conditions:

For all $x \in Y$, $\dot{x} \longrightarrow x$ (where $\dot{x}$ denotes the ultrafilter generated by $\{x\}$);

$\mathfrak{G} \supset \mathfrak{F} \longrightarrow x$ implies $\mathfrak{G} \longrightarrow x$.

A function $f : X \longrightarrow Y$ between convergence spaces is <u>continuous</u> iff $\mathfrak{F} \xrightarrow[X]{} x$ implies $f(\mathfrak{F}) \xrightarrow[Y]{} f(x)$, where $f(\mathfrak{F})$ denotes the filter on Y generated by $\{ f(F) \mid F \in \mathfrak{F} \}$. We denote the category of convergence spaces and continuous functions by <u>Conv</u>.

A convergence space is called

<u>localized</u> iff $\mathfrak{F} \longrightarrow x$ implies $\mathfrak{F} \cap \dot{x} \longrightarrow x$,

<u>limit space</u> iff $\mathfrak{F} \longrightarrow x$ and $\mathfrak{G} \longrightarrow x$ implies $\mathfrak{F} \cap \mathfrak{G} \longrightarrow x$,

<u>pseudotopological</u> iff $\mathfrak{F} \longrightarrow x$ whenever $\mathfrak{U} \longrightarrow x$ for every ultrafilter $\mathfrak{U} \supset \mathfrak{F}$, and

<u>pretopological</u> iff for all $x \in Y$, $\mathfrak{V}(x) = \bigcap \{ \mathfrak{F} \mid \mathfrak{F} \longrightarrow x \}$ converges to x. The corresponding subcategories of <u>Conv</u> are denoted by <u>LConv</u>, <u>Lim</u>, <u>PsT</u>, and <u>PrT</u>, respectively.

A localized convergence space is said to fulfil the R_0-axiom if

$\mathfrak{F} \cap \dot{x} \longrightarrow y$ implies $\mathfrak{F} \longrightarrow x$

(i.e., whenever $\mathfrak{F} \cap \dot{x}$ is convergent, then $\mathfrak{F}$ converges to x). The subcategory of localized R_0-spaces is denoted by R_0<u>LConv</u>; R_0<u>Lim</u> etc. are defined accordingly.

Finally, we denote by <u>ConsConv</u> the subcategory of <u>Conv</u> which consists of those convergence spaces where the same filters converge to every point (i.e., the assigning function is constant).

All categories of Definition 3.1 are topological. Moreover, <u>Conv</u> $\supset$ <u>LConv</u> $\supset$ <u>Lim</u> $\supset$ <u>PsT</u> $\supset$ <u>PrT</u> $\supset$ <u>Top</u>, <u>Conv</u> $\supset$ <u>ConsConv</u>, <u>LConv</u> $\supset$ R_0<u>LConv</u>, and in these chains, every category is properly contained as a bireflective subcategory in the preceding one(s). <u>LConv</u> is also bicoreflective in <u>Conv</u>. The category <u>ConsConv</u> is isomorphic to the category <u>Grill</u> of grill-determined prenearness spaces [Ro 75], [BHR 76] and Katětov's filter-merotopic spaces [Ka 65]. Similarly, R_0<u>LConv</u> is isomorphic to the category <u>LGrill</u> of localized grill-determined spaces and Katětov's localized filter-merotopic spaces. <u>LGrill</u> is bicoreflective in <u>Grill</u>.

<u>3.2</u> <u>Example</u>. By 2.2, the category <u>Conv</u> is hereditary. 2.3 suggests

what the one-point-extensions should look like (cf. the slightly erroneous construction for Lim in [Pe 73: Proposition 8]). For any convergence space Y, the one-point-extension [Y] can be described as follows: All filters on [Y] converge to ∞_Y, while for $x \in Y$, $\mathfrak{F} \xrightarrow[{[Y]}]{} x$ iff $\mathfrak{F} \supset \overline{\mathfrak{G}} \cap \dot{\infty}_Y$ for some $\mathfrak{G} \xrightarrow[Y]{} x$ (or, equivalently, $\mathfrak{F} \xrightarrow[{[Y]}]{} x$ iff $\mathfrak{F} \restriction Y \xrightarrow[Y]{} x$, for $x \in Y$). (Here, for filters $\mathfrak{F}$ on [Y] and $\mathfrak{G}$ on Y, $\mathfrak{F} \restriction Y$ denotes the restriction of $\mathfrak{F}$ to Y and $\overline{\mathfrak{G}}$ the filter on [Y] generated by $\mathfrak{G}$.)

3.3 Example. (1) LConv is hereditary. This can be concluded from 3.2 and 2.8, or with 2.7, since a simple computation shows that LConv is closed under formation of one-point-extensions in Conv.

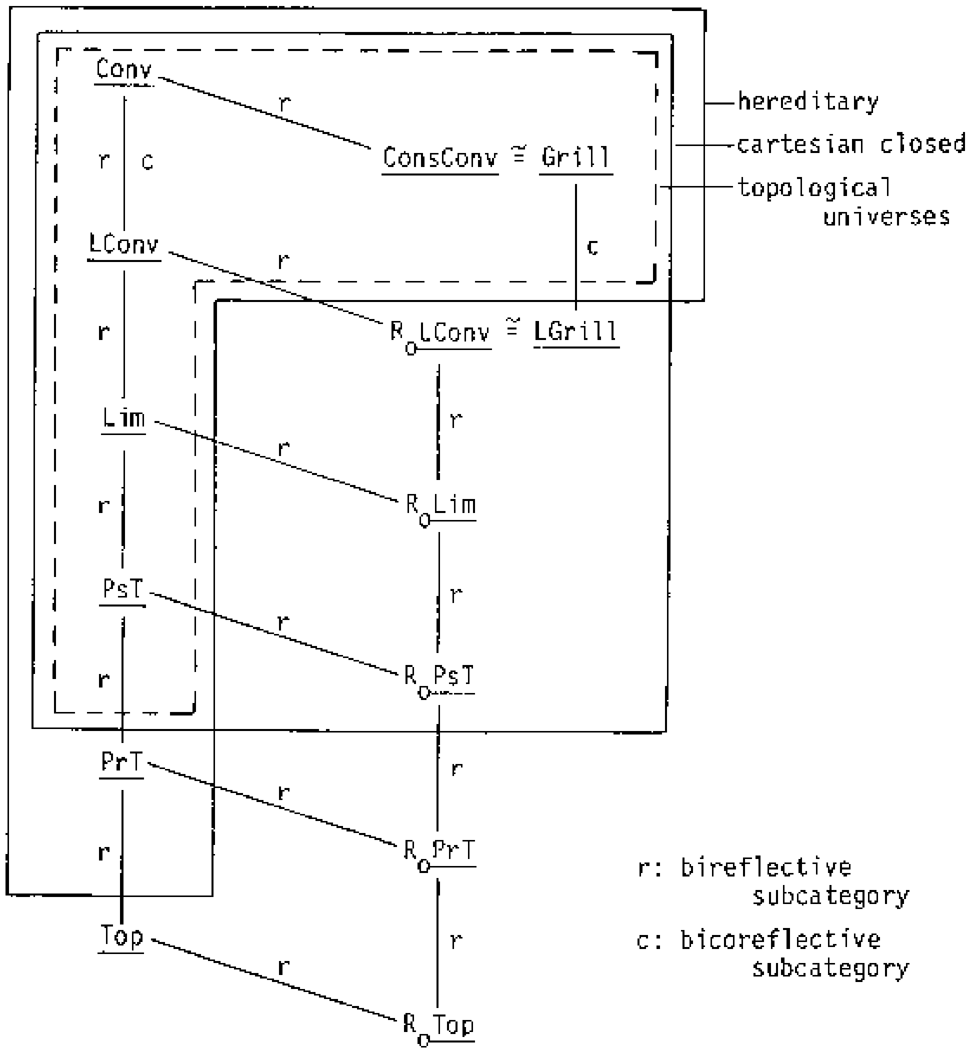
(2) Again by looking at one-point-extensions, we see that ConsConv is hereditary. For $Y \in$ ConsConv, [Y] is given by: $\mathfrak{F} \xrightarrow[{[Y]}]{} x$ iff $\mathfrak{F} \supset \overline{\mathfrak{G}} \cap \dot{\infty}_Y$ for some $\mathfrak{G} \xrightarrow[Y]{} y$. Notice that the one-point-extensions in ConsConv are not formed as in Conv.

3.4 Example. (1) Recall that the categories Top, PrT, PsT, and Lim are finally dense in LConv. Hence the simplest possibility to prove or disprove heredity is to check whether the category in question is closed under formation of one-point-extensions in Conv (2.7, 3.3(1)). In this way we can show that Lim and PsT are hereditary (which is known from [Pe 77]).– To prove that PrT is hereditary [He 86] is even simpler since the pretopological space $\mathfrak{Z}$ defined by $|\mathfrak{Z}| = \{0,1,2\}$, $\mathfrak{U}(0) = \overline{\{0,1,2\}} = \mathfrak{U}(2)$, $\mathfrak{U}(1) = \overline{\{1,2\}}$ constitutes an initially dense subclass of PrT [Bo 75: proof of II.2.1], and we have only to check that $[\mathfrak{Z}]_{Conv} \in$ PrT.– In the same way we see that Top is not hereditary: Consider the Sierpinski space $\mathcal{Z} = (\{0,1\},\{\emptyset,\{1\},\{0,1\}\})$. Then $[\mathcal{Z}]_{Conv} = \mathfrak{Z} \notin$ Top.

(2) A differnt possibility to see that Top is not hereditary is the application of 2.3: $\{\emptyset,\{1\},\{0,1,2\}\}$ and $\{\emptyset,\{1,2\},\{0,1,2\}\}$ are maximal topologies on $\{0,1,2\}$ which make $\mathcal{Z}$ a subspace; but there is no greatest topology with this property.

3.5 Example. For every $Y \in R_0$LConv, define Y^+ by: $|Y^+| = |Y| \cup \{\infty_Y\}$; for $x \in Y$, $\mathfrak{F} \xrightarrow[Y^+]{} x$ iff $\mathfrak{F} \restriction Y \xrightarrow[Y]{} x$; and $\mathfrak{F} \xrightarrow[Y^+]{} \infty_Y$ iff $\mathfrak{F} = \mathfrak{G} \cap \dot{y}$ with $y \in Y$ implies $\mathfrak{G} \restriction Y \xrightarrow[Y]{} y$. Then Y^+ is the greatest localized R_0-convergence space which has Y as a subspace. However, R_0LConv is not hereditary: It can be shown that for the discrete space N whose underlying set is the set of natural numbers, N^+ is not a one-point-extension of N; in particular, $N^+ < [N]$.— Indeed, one can prove that none of the categories of R_0-spaces resulting from Definition 3.1 is hereditary.

The behavior of the convergence categories considered in this section towards heredity is summed up in the following diagram. For completeness, we added information about cartesian closedness and topological universes.



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